
II. *A Method for determining the Number of impoſſible Roots in adfected Æquations. By Mr.* George Campbell.

LEMMA I.

IN every adfected quadratick Æquation $ax^2 - Bx + A = 0$, whoſe Roots are real, a fourth Part of the Square of the Coefficient of the ſecond Term is greater than the Rectangle under the Coefficient of the firſt Term and the abſolute Number or $\frac{1}{4}B^2 > a \times A$; and *vice verſa* if $\frac{1}{4}B^2 > a \times A$, the Roots of the Æquation $ax^2 - Bx + A = 0$, will be real. But if $\frac{1}{4}B^2 < a \times A$, the Roots will be impoſſible. This is evident from the Roots of the Æquation being $\frac{\frac{1}{2}B + \sqrt{\frac{1}{4}B^2 - a \times A}}{a}$, $\frac{\frac{1}{2}B - \sqrt{\frac{1}{4}B^2 - a \times A}}{a}$.

LEMMA II.

Whatever be the Number of impoſſible Roots in the Æquation $x^n - Bx^{n-1} + Cx^{n-2} - Dx^{n-3} + \&c. \pm dx^3 \mp cx^2 \pm bx \mp A = 0$, there are juſt as many in the Æquation $Ax^n - bx^{n-1} + cx^{n-2} - dx^{n-3} + \&c. \pm Dx^3 \mp Cx^2 \pm Bx \mp 1 = 0$. For the Roots of the laſt Æquation are the Reciprocals of thoſe of the firſt, as is evident from common Algebra. Let the Roots of the biquadratick Æquation $x^4 - Bx^3 + Cx^2 - Dx + A = 0$ be a, b, c, d, whereof let c, d be impoſſible, then the Roots of the Æquation

$Ax^4 - \mathcal{D}x^3 + Cx^2 - Bx + 1 = 0$ will be $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, and therefore two of them to wit $\frac{1}{c}, \frac{1}{d}$ impoſſible.

LEMMA III.

In every Æquation $x^n - Bx^{n-1} + Cx^{n-2} - \mathcal{D}x^{n-3} + Ex^{n-4} - \&c. \pm ex^4 \mp dx^3 \pm cx^2 \mp bx \pm A = 0$, all whoſe Roots are real, if each Term be multiply'd by the Index of x in that Term, and each Product be divided by x, the reſulting Æquation $nx^{n-1} - \overline{n-1}\,Bx^{n-2} + \overline{n-2}\,Cx^{n-3} - \overline{n-3}\,\mathcal{D}x^{n-4} + \overline{n-4}\,Ex^{n-5} - \&c. \pm 4ex^3 \mp 3dx^2 \pm 2cx \mp b = 0$ ſhall have all its Roots real. Thus if all the Roots of the Æquation $x^4 - Bx^3 + Cx^2 - \mathcal{D}x + A = 0$ be real, then all the Roots of the Æquation $4x^3 - 3Bx^2 + 2Cx - \mathcal{D} = 0$ will alſo be real. This Lemma doth not hold converſly, for there are an Infinity of Caſes where all the Roots of the Æquation $nx^{n-1} - \overline{n-1}\,Bx^{n-2} + \overline{n-2}\,Cx^{n-3} - \overline{n-3}\,\mathcal{D}x^{n-4} + \&c. \pm 3dx^2 \mp 2cx \pm b = 0$ are real, at the ſame Time ſome or perhaps all the Roots of the Æquation $x^n - Bx^{n-1} + Cx^{n-2} - \mathcal{D}x^{n-3} + \&c. \pm dx^3 \mp cx^2 \pm bx \mp A = 0$ are impoſſible: But whatever be the Number of impoſſible Roots in the Æquation $nx^{n-1} - \overline{n-1}\,Bx^{n-2} + \overline{n-2}\,Cx^{n-3} - \&c. \pm 2cx \mp b = 0$, there are at leaſt as many in the Æquation $x^n - Bx^{n-1} + Cx^{n-2}\ \&c. \pm cx^2 \mp bx \pm A = 0$. Thus all the Roots of the Æquation $4x^3 - 3Bx^2 + 2Cx - \mathcal{D} = 0$ may be real, and yet two or perhaps all the four Roots of the Æquation $x^4 - Bx^3 + Cx^2 - Dx + A = 0$ may be impoſſible, but if two of the Roots of the Æquation $4x^3 - 3Bx^2 + 2Cx - D = 0$ be impoſſible, there muſt be at leaſt two impoſſible Roots in the Æquation $x^4 - Bx^3 + Cx^2 - Dx + A = 0$. All this hath been demonſtrated by Algebraical Writers, particularly by Mr. *Reyneau* in his *Analyſe Demontré*, and is eaſily made evident by the Method of the *Maxima* and *Minima*.

COROLARY. Let all the Roots of the Æquation $x^n - Bx^{n-1} + Cx^{n-2} - Dx^{n-3} + Ex^{n-4} - Fx^{n-5} + \&c. \pm fx^5 \mp ex^4 \pm dx^3 \mp cx^2 \pm bx \mp A = 0$ be real, and by this Lemma all the Roots of the Æquation $nx^{n-1} - \overline{n-1}\,Bx^{n-2} + \overline{n-2}\,Cx^{n-3} - \overline{n-3}\,Dx^{n-4} + \overline{n-4}\,Ex^{n-5} - \overline{n-5}\,Fx^{n-6} + \&c. \pm 5fx^4 \mp 4ex^3 \pm 3dx^2 \mp 2cx \pm b = 0$ will be real, and therefore (by the ſame Lemma) all the Roots of the Æquation $n \times \overline{n-1}\,x^{n-2} - \overline{n-1} \times \overline{n-2}\,Bx^{n-3} + \overline{n-2} \times \overline{n-3}\,Cx^{n-4} - \overline{n-3} \times \overline{n-4}\,Dx^{n-5} + \overline{n-4} \times \overline{n-5}\,Ex^{n-6} - \overline{n-5} \times \overline{n-6}\,Fx^{n-7} + \&c. \pm 20fx^3 \mp 12ex^2 \pm 6dx \mp 2c = 0$ or (dividing all by 2) of $n \times \frac{\overline{n-1}}{2}x^{n-2} - \overline{n-1} \times \frac{\overline{n-2}}{2}Bx^{n-3} + \overline{n-2} \times \frac{\overline{n-3}}{2}Cx^{n-4} - \&c. \pm 10fx^3 \mp 6ex^2 \pm 3dx \mp c = 0$ will be real. After the ſame Manner all the Roots of the Æquation $n \times \frac{\overline{n-1}}{2} \times \frac{\overline{n-2}}{3}x^{n-3} - \overline{n-1} \times \frac{\overline{n-2}}{2} \times \frac{\overline{n-3}}{3}Bx^{n-4} +$

$Bx^{n-4}+\overline{n-2}\times\frac{\overline{n-3}}{2}\times\frac{\overline{n-4}}{3}Cx^{n-5}-$ &c. $\pm 10fx^2\mp 4ex\pm d=0$ will be real; and thus we may deſcend until we arrive at the quadratick Æquation $n\times\frac{\overline{n-1}}{2}x^2-\overline{n-1}Bx+C=0$. The ſame Æquations do aſcend thus $n\times\frac{\overline{n-1}}{2}x^2-\overline{n-1}Bx+C=0$, $n\times\frac{\overline{n-1}}{2}\times\frac{\overline{n-2}}{3}x^3-\overline{n-1}\times\frac{\overline{n-2}}{2}Bx^2+\overline{n-2}Cx-D=0$, $n\times\frac{\overline{n-1}}{2}\times\frac{\overline{n-2}}{3}\times\frac{\overline{n-3}}{4}x^4$ $\overline{n-1}\times\frac{\overline{n-2}}{2}\times\frac{\overline{n-3}}{3}Bx^3+\overline{n-2}\times\frac{\overline{n-3}}{2}\times Cx^2-\overline{n-3}Dx+E=0$, $n\times\frac{\overline{n-1}}{2}\times\frac{\overline{n-2}}{3}\times\frac{\overline{n-3}}{4}\times\frac{\overline{n-4}}{5}x^5-\overline{n-1}\times\frac{\overline{n-2}}{2}\times\frac{\overline{n-3}}{3}\times\frac{\overline{n-4}}{4}Bx^4+\overline{n-2}\times\frac{\overline{n-3}}{2}\times\frac{\overline{n-4}}{3}Cx^3-\overline{n-3}\times\frac{\overline{n-4}}{2}Dx^2+\overline{n-4}Ex-F=0$, and ſo on. Let M repreſent any of the Coefficients of the Æquation $x^n-Bx^{n-1}+Cx^{n-2}-Dx^{n-3}+Ex^{n-4}-$ &c. $\pm A=0$, and let L N be the adjacent Coefficients, let M be the Exponent of the Coefficient M: By the Exponent of a Coefficient I mean the Number which expreſſeth the Place which it hath among the Coefficients, thus if M repreſent the Coefficient E (and therefore $L = D$ and $N = F$) then $m = 4$. It will be eaſy to ſee, that, amongſt the foregoing aſcending Æquations, that which hath its abſolute Number N will be $n \times \frac{\overline{n-1}}{2} \times \frac{\overline{n-2}}{3} \times$ &c. $\frac{\overline{n-m}}{m+1} x^{m+1} - \overline{n-1} \times \frac{\overline{n-2}}{2} \times$ &c. $\frac{\overline{n-m}}{m} B x^{m} + \overline{n-2} \times$ &c. $\frac{\overline{n-m}}{m-1} C x^{m-1} -$ &c. $\pm \overline{n-m+1} \times \frac{\overline{n-m}}{2} L x^{2} \mp \overline{n-m} M x \pm N = 0$, all whoſe Roots are real when all the Roots of the Æquation $x^{n} - B x^{n-1} + C x^{n-2} -$ &c. $\pm A = 0$ are real. Let $N = F$ and therefore $M = E$, $L = D$ and $m = 4$, then that of the aſcending Æquations whoſe abſolute Number is F, will be $n \times \frac{\overline{n-1}}{2} \times \frac{\overline{n-2}}{3} \times \frac{\overline{n-3}}{4} \times \frac{\overline{n-4}}{5} x^{5} - \overline{n-1} \times \frac{\overline{n-2}}{2} \times \frac{\overline{n-3}}{3} \times \frac{\overline{n-4}}{4} B x^{4} + \overline{n-2} \times \frac{\overline{n-3}}{2} \times \frac{\overline{n-4}}{3} C x^{3} - \overline{n-3} \times \frac{\overline{n-4}}{2} D x^{2} + \overline{n-4} E x - F = 0.$

PROPOSITION I.

Let $x^n - Bx^{n-1} + Cx^{n-2} - Dx^{n-3} + Ex^{n-4} - \&c. \pm ex^4 \mp dx^3 \pm cx^2 \mp bx \pm A = 0$ be an Æquation of any Dimenſions all whoſe Roots are real, let M be any Coefficient of this Æquation, L, N the adjacent Coefficients, and m the Exponent of M. Then the Square of any Coefficient M multiply'd by the Fraction $\frac{m \times \overline{n-m}}{\overline{m+1} \times \overline{n-m+1}}$ will always exceed the Rectangle under the adjacent Coefficients $L \times N$. Thus in the Æquation $x^4 - Bx^3 + Cx^2 - Dx + A = 0$, where $n = 4$, making $M = C$ and therefore $L = B$, $N = D$, and $m = 2$, then $\frac{2 \times \overline{4-2}}{\overline{2+1} \times \overline{4-2+1}} \times C^2$ or $\frac{4}{9}C^2$ will exceed $B \times D$ providing all the Roots of the Æquation be real.

Becauſe (by Lem. 3.) the Roots of the quadratick Æquation $n \times \frac{\overline{n-1}}{2} x^2 - \overline{n-1}\, Bx + C = 0$, are real, therefore (by Lem. 1.) $\frac{1}{4}\overline{n-1}|^2 \times B^2$ muſt be greater than $n \times \frac{\overline{n-1}}{2} \times C$ and (dividing both by $n \times \frac{\overline{n-1}}{2}$) $\frac{\overline{n-1}}{2n} \times B^2$ greater than $1 \times C$. Therefore in the Æquation $x^n - Bx^{n-1} + Cx^{n-2} - Dx^{n-3} + \&c. \pm A = 0$ of the n Degree, all whoſe Roots are real, the Square of B the Coefficient of the ſecond

Term, multiply'd by the Fraction $\frac{n-1}{2n}$ is greater than $1 \times C$ the Rectangle under the adjacent Coefficients. But (by Lem. 2.) all the Roots of the Æquation $Ax^n - bx^{n-1} + cx^{n-2} - \&c. \pm Cx^2 \mp Bx \pm 1 = 0$ or (dividing by A) of $x^n - \frac{b}{A}x^{n-1} + \frac{c}{A}x^{n-2} - \&c. \pm \frac{C}{A}x^2 \mp \frac{B}{A}x \pm \frac{1}{A} = 0$ are real, therefore (from what hath been juſt now ſaid) $\frac{\overline{n-1}}{2n} \times \frac{b^2}{A^2}$ muſt be greater than $1 \times \frac{c}{A}$ and conſequently $\frac{\overline{n-1}}{2n} \times b^2$ greater than $c \times A$. Therefore in an Æquation $x^n - Bx^{n-1} + Cx^{n-2} - \&c. \pm cx^2 \mp bx \pm A = 0$, of the n Degree, all whoſe Roots are real, the Square of the Coefficient of x multiply'd by the Fraction $\frac{n-1}{2n}$ is greater than the Rectangle under the Coefficient of x^2 and the abſolute Number. But by Cor. Lem. 3. all the Roots of the Æquation $n \times \frac{\overline{n-1}}{2} \times \frac{\overline{n-2}}{3} \times \&c. \times \frac{\overline{n-m}}{m+1}x^{m+1} - \overline{n-1} \times \frac{n-2}{2} \times \&c. \times \frac{\overline{n-m}}{m}Bx^m + \overline{n-2} \times \&c.$ $\frac{\overline{n-m}}{m-1}Cx^{m-1}\ \&c. \pm \overline{n-m+1} \times \frac{\overline{n-m}}{2} \times Lx^2 \mp$

$\overline{n-m}\,Mx \pm N = 0$ are real, therefore (ſeeing this Æquation is of the $m+1$ Degree) the Square of $\overline{n-m} \times M$ multiply'd by the Fraction $\frac{\overline{m+1}-1}{2 \times \overline{m+1}}$ will be greater than the Rectangle under $\overline{n-m+1} \times \frac{\overline{n-m}}{2} \times L$ and N, that is $\frac{m}{2 \times \overline{m+1}} \times \overline{n-m}|^2 \times M^2$ will be greater than $\overline{n-m+1} \times \frac{\overline{n-m}}{2} \times L \times N$ and therefore (dividing both by $\overline{n-m+1} \times \frac{\overline{n-m}}{2}$) $\frac{m \times \overline{n-m}}{\overline{m+1} \times \overline{n-m+1}} \times M^2$ greater than $L \times N$.

COROLARY. Make a Series of Fractions $\frac{n}{1}$, $\frac{n-1}{2}$, $\frac{n-2}{3}$, $\frac{n-3}{4}$, &c. unto $\frac{1}{n}$ whoſe Denominators are Numbers going on in the Progreſſion 1, 2, 3, 4, &c. unto the Number n which is the Dimenſions of the Æquation $x^n - Bx^{n-1} + Cx^{n-2} -$ &c. $\pm A = 0$, and whoſe Numerators are the ſame Progreſſion inverted. Divide the ſecond of theſe Fractions by the firſt, the third by the ſecond, the fourth by the third, and ſo on, and place the Fractions which reſult from this Diviſion above the middle Terms of the Æquation, thus $x^n - \overset{\frac{n-1}{2n}}{B}x^{n-1} + \overset{\frac{2 \times \overline{n-2}}{3 \times \overline{n-1}}}{C}x^{n-2} - Dx^{n-3} +$

$\overset{\frac{3\times\overline{n-3}}{4\times\overline{n-2}}}{D}x^{n-3}+\overset{\frac{4\times\overline{n-4}}{5\times\overline{n-3}}}{E}x^{n-4}-$ &c. $\pm A=0$. Then if all the Roots of the Æquation are real, the Square of any Coefficient multiply'd by the Fraction which ſtands above, will be greater than the Rectangle under the adjacent Coefficients. This Corolary doth not hold converſly, for there are an Infinity of Æquations in which the Square of each Coefficient multiply'd by the Fraction above it, may be greater than the Rectangle under the adjacent Coefficients, and notwithſtanding ſome or perhaps all of the Roots may be impoſſible. Therefore when the Square of a Coefficient multiply'd by the Fraction above, is greater than the Rectangle under the adjacent Coefficients, from this Circumſtance nothing can be determined as to the Poſſibility or Impoſſibility of the Roots of the Æquation: But when the Square of a Coefficient multiply'd by the Fraction above it, is leſs than the Rectangle under the adjacent Coefficients, it is a certain Indication of two impoſſible Roots. From what hath been ſaid, is immediately deduced the Demonſtration of that Rule which the moſt illuſtrious *Newton* gives for determining the Number of impoſſible Roots in any given Æquation.

SCHOLIUM.

Let the Roots of the Æquation $x^n-Bx^{n-1}+Cx^{n-2}-Dx^{n-3}+Ex^{n-4}-Fx^{n-5}+$ &c. $\pm A=0$ (with their Signs) be repreſented by the Letters a, b, c, d, e, f, g, &c. then (as is commonly known) B will be the Sum of all the Roots or $=a+b+c+d+e+f+$ &c. C the Sum of the Products of all the Pairs of Roots or $= ab + ac + ad + af + ag +$ &c. D the Sum of the Products of all the Ternaryes of Roots or $= abc + abd + abe + abf + abg +$ &c. $E = abcd + abce + abcf + abeg +$ &c. $F = abcde + abcdf + abcdg + bcdef +$ &c. and ſo on. Let (as in this Propoſition) M repreſent any of theſe Coefficients, L, N the adjacent Coefficients, and m the Exponent of M; let Z repreſent the Sum of the Squares of all the poſſible Differences between the Terms of the Coefficient M, let α be the Sum of all thoſe of the foreſaid Squares whoſe Terms differ by one Letter, β the Sum of all thoſe Squares whoſe Terms differ by two Letters, γ the Sum of thoſe Squares whoſe Terms differ by three Letters, δ the Sum of thoſe Squares whoſe Terms differ by four Letters and ſo on. Thus if $M = F = abcde + abcdf + abcdg +$ &c. then $Z = \overline{abcde - abcdf}|^2 + \overline{abcde - abcdg}|^2 + \overline{abcde - abcfg}|^2 + \overline{bcdef - abfgh}|^2 +$ &c. $\alpha = \overline{abcde - abcdf}|^2 + \overline{abcde - abcdg}|^2 + \overline{abcde - abcdh}|^2 + \overline{bcdef - bcdeg}|^2 +$ &c. $\beta = \overline{abcde - abcfg}|^2 + \overline{abcde - abcfh}|^2 + \overline{bcdef - acdfh}|^2 +$ &c. $\gamma = \overline{abcde - abfgh}|^2 + \overline{abcdf - abegh}|^2 +$ &c. $\delta = \overline{abcde - afghk}|^2 + \overline{acdfg - abehk}|^2 +$ &c. This being laid down I ſay that the Square of any Coefficient M multiply'd by the Fraction $\dfrac{m \times \overline{n - m}}{\overline{m + 1} \times \overline{n - m + 1}}$ exceeds the Rectangle under the adjacent Coefficients $L \times N$ by $\overline{n + 1} \times Z$

$\frac{\overline{n+1}\times Z}{\overline{m+1}\times\overline{n-m+1}} - \frac{1}{2}\alpha - \frac{1}{3}\beta - \frac{1}{4}\gamma - \frac{1}{5}\delta - \&c.$ The Series $- \frac{1}{2}\alpha - \frac{1}{3}\beta - \frac{1}{4}\gamma -$ &c. muſt conſiſt of m Number of Terms.

Let the Æquation be $x^5 - Bx^4 + Cx^3 - Dx^2 + Ex - A = 0$, whoſe Roots let be a, b, c, d, e, in which Caſe $n = 5$. Let $M = B = a + b + c + d + e$, then $L = 1$, $N = G$, $m = 1$, $Z = \overline{a-b}|^2 + \overline{a-c}|^2 + \overline{a-d}|^2 + \overline{a-e}|^2 + \overline{b-c}|^2 + \&c. = \alpha$; therefore $\frac{1\times\overline{5-1}}{\overline{1+1}\times\overline{5-1+1}}\times B^2$ or $\frac{2}{5}B^2$ exceeds $1\times C$ by $\frac{\overline{5+1}\times Z}{\overline{1+1}\times\overline{5-1+1}} - \frac{1}{2}\alpha = \frac{3}{5}Z - \frac{1}{2}\alpha =$ (becauſe $Z = \alpha$) $\frac{1}{10}Z = \frac{1}{10}\overline{a-b}|^2 + \frac{1}{10}\times\overline{a-c}|^2 + \frac{1}{10}\overline{a-d}|^2 +$ &c. which is always a poſitive Number when the Roots a, b, c, d, e are real, poſitive or negative Numbers. Let $M = C = ab + ac + ad + ae + bc +$ &c. then $L = B$, $N = D$, $m = 2$, $Z = \overline{ab-ac}|^2 + \overline{ab-ad}|^2 + \overline{ab-cd}|^2 + \overline{ab-de}|^2 + \&c.$ $\alpha = \overline{ab-ac}|^2 + \overline{ab-ad}|^2 + \overline{ab-ae}|^2 + \&c.$ $\beta = \overline{ab-cd}|^2 + \overline{ab-ce}|^2 + \overline{ab-de}|^2 + \&c.$ therefore $\frac{2\times\overline{5-2}}{\overline{2+1}\times\overline{5-2+1}}\times C^2$

or $\frac{1}{2}C^2$ ſurpaſſeth $B\times\mathcal{D}$ by $\frac{\overline{5+1}\times Z}{\overline{2+1}\times\overline{5-2+1}}$ $-\frac{1}{2}\alpha-\frac{1}{3}\beta=$ (becauſe $Z=\alpha+\beta$) $=\frac{1}{6}$ $\beta=\frac{1}{6}\times\overline{a\,b-c\,d}|^2+\frac{1}{6}\overline{a\,b-c\,e}|^2+\frac{1}{6}\times$ $\overline{a\,b-d\,e}|^2+$ &c. which is always a poſitive Number when the Roots a, b, c, d, e are real Numbers, poſitive or negative. Let $M=\mathcal{D}=a\,b\,c+a\,b\,d+a\,b\,e+a\,c\,d+a\,c\,e+$ &c. then $L=C$, $N=E$, $m=3$, $Z=\overline{a\,b\,c-a\,b\,d}|^2+\overline{a\,b\,c-a\,b\,e}|^2+\overline{a\,b\,c-a\,d\,e}|^2+$ &c. $\alpha=\overline{a\,b\,c-a\,b\,d}|^2+\overline{a\,b\,c-a\,b\,e}|^2+\overline{a\,b\,c-a\,c\,d}|^2+$ &c. $\beta=\overline{a\,b\,c-a\,d\,e}|^2+\overline{a\,b\,c-c\,d\,e}|^2+\overline{a\,b\,c-b\,d\,e}|^2+$ &c. $\gamma=0$, therefore $\frac{3\times\overline{5-3}}{\overline{3+1}\times\overline{5-3+1}}\times$ $\mathcal{D}^2$ or $\frac{1}{2}\mathcal{D}^2$ exceeds $C\times E$ by $\frac{\overline{5+1}}{\overline{3+1}\times\overline{5-3+1}}\times$ $Z-\frac{1}{2}\alpha-\frac{1}{3}\beta=$ (becauſe $Z=\alpha+\beta$) $=\frac{1}{6}\times$ $\beta=\frac{1}{6}\times\overline{a\,b\,c-a\,d\,e}|^2+\frac{1}{6}\times\overline{a\,b\,c-c\,d\,e}|^2+\frac{1}{6}$ $\overline{a\,b\,c-b\,d\,e}|^2+$ &c. which is a poſitive Number when the Roots are real Numbers. Let $M=E=a\,b\,c\,d+a\,b\,c\,e+a\,b\,d\,e+b\,c\,d\,e+$ &c. then $L=\mathcal{D}$, $N=A$, $m=4$, $Z=\overline{a\,b\,c\,d-a\,b\,c\,e}|^2+\overline{a\,b\,c\,d-b\,c\,d\,e}|^2+\overline{a\,b\,c\,d-a\,c\,d\,e}|^2+$ &c. $=\alpha$, $\beta=$

$\beta = 0 = \gamma = \delta$, therefore $\frac{4\times\overline{5-4}}{\overline{4+1}\times\overline{5-4+1}}\times E^2$ or $\frac{2}{5}E^2$ exceeds $D\times A$ by $\frac{\overline{5+1}}{\overline{4+1}\times\overline{5-4+1}}\times Z - \frac{1}{2}\alpha = \frac{3}{5}Z - \frac{1}{2}\alpha = \frac{1}{10}Z = \frac{1}{10}\times\overline{abcd-abce}|^2 + \frac{1}{10}\times\overline{abcd-bcde}|^2 +$ &c. which is a poſitive Number when the Roots are real Numbers.

PROPOSITION II.

Let $x^n - Bx^{n-1} + Cx^{n-2} - Dx^{n-3} + Ex^{n-4} -$ &c. $\pm A = 0$ be an Æquation of any Degree, whoſe Roots with their Signs let be expreſſed by the Letters a, b, c, d, e, f, &c. let M repreſent any Coefficient of this Æquation, L, N the Coefficients adjacent to M; K, O the Coefficients adjacent to L, N; I, P thoſe adjacent to K, O; H, Q thoſe adjacent to I, P, and ſo on. Let m repreſent the Exponent of M and let Z (as in the preceeding Propoſition) repreſent the Sum of the Squares of all the poſſible Differences between the Terms of the Coefficient M. Then the Product of the Square of any Coefficient M multiply'd by the Fraction $\frac{1}{2}\times\overline{1-\frac{1}{n\times\frac{\overline{n-1}}{2}\times\frac{\overline{n-2}}{3}\times\&c.\times\frac{\overline{n-m+1}}{m}}}$ doth always exceed $L\times N - K\times O + I\times P - H\times Q +$ &c. by $\frac{\frac{1}{2}Z}{n\times\overline{\frac{n-1}{2}}\times\overline{\frac{n-2}{3}}\times \&c.\times\overline{\frac{n-m+1}{m}}}$ which is always a poſitive Number, when the Roots a, b, c, d, e &c. are real Numbers poſitive or negative. Let the Æquation be of the ſeventh Degree or $x^7 - Bx^6 + Cx^5 - Dx^4 + Ex^3 - Fx^2 + Gx - A = 0$, whoſe Roots let be a, b, c, d, e, f, g, in which Caſe $n = 7$. Let $M = E = abcd + abce + abcf + abcg + bcde +$ &c. then $m = 4$, $L = -D$, $N = -F$, $K = C$, $O = G$, $I = -B$, $P = -A$, $Z = \overline{abcd - abce}|^2 + \overline{abcd - abcf}|^2 + \overline{abcd - abcg}|^2 +$ &c. Therefore $\frac{1}{2}\times\overline{1 - \frac{1}{7\times\frac{6}{2}\times\frac{5}{3}\times\frac{4}{4}}}\times E^2$ or $\frac{17}{35}E^2$ exceeds $D\times F - C\times G + B\times A$ by $\frac{\frac{1}{2}Z}{7\times\frac{6}{2}\times\frac{5}{3}\times\frac{4}{4}}$ or $\frac{Z}{70} = \frac{1}{70}\times\overline{abcd - abce}|^2 + \frac{1}{70}\times\overline{abcd - abcf}|^2 +$ &c.

From this Propoſition, is deduced the following Rule for determining the Number of impoſſible Roots in any given Æquation. From each of the Unciæ of the middle Terms of that Power of a Binomial, whoſe Index is the Dimenſions of the propoſed Æquation, ſubtract Unity, then divide each Remainder by twice the Correſpondent *Vncia*, and ſet the Fractions which reſult from this Diviſion, above the middle Terms of the given Æquation. And under any of the middle Terms if its Square multiplyed by the Fraction ſtanding above it, be greater than the Rectangle under the immediately adjacent Terms, ***Minus*** the Rectangle under the next adjacent Terms, ***Plus*** the Rectangle under the Terms then next adjacent — *&c.* place the Sign $+$, but if it be leſs, place the Sign $-$. And under the firſt and laſt Term place $+$. And there will be at leaſt as many impoſſible Roots, as there are Changes in the Series of the under-written Signs from $+$ to $-$, or from $-$ to $+$. Let it be required to determine the Number of impoſſible Roots in the Æquation $x^7 - 5x^6 + 15x^5 - 23x^4 + 18x^3 + 10x^2 - 28x + 24 = 0$. The *Vnciæ* of the middle Terms of the 7th Power of a Binomial are 7, 21, 35, 35, 21, 7, from which ſubtracting Unity, and dividing each of the Remainders by twice the correſpondent *Vncia*, the Quotients will be $\frac{6}{14}$, $\frac{20}{42}$, $\frac{34}{70}$, $\frac{34}{70}$, $\frac{20}{42}$, $\frac{6}{14}$ or $\frac{3}{7}$, $\frac{10}{21}$, $\frac{17}{35}$, $\frac{17}{35}$, $\frac{10}{21}$, $\frac{3}{7}$ which Fractions place above the middle Terms of the Æquation, has $\underset{+}{x^7} - \underset{-}{\overset{\frac{3}{7}}{5}}x^6 + \underset{+}{\overset{\frac{10}{21}}{15}}x^5 -$

$\underset{-}{\overset{\frac{17}{35}}{23x^4}}+\underset{+}{\overset{\frac{17}{35}}{18x^3}}+\underset{-}{\overset{\frac{10}{21}}{10x^2}}-\underset{+}{\overset{\frac{3}{7}}{28x}}+\underset{+}{24}=0.$ Then becauſe the Square of $-5x^6$ multiply'd into the Fraction over its Head $\frac{3}{7}$, to wit $\frac{75}{7}x^{12}$ is leſs than $x^7\times15x^5$ or $15x^{12}$ I place the Sign — under the Term $5x^6$. Becauſe the Square of $15x^5$ multiply'd by the Fraction over its Head $\frac{10}{21}$ to wit $\frac{705}{7}x^{10}$ is greater than $\overline{-5x^6\times-23x^4}-\overline{x^7\times18x^3}=97x^{10}$ I place the Sign + under the Term $15x^5$. Seeing $\frac{8993}{35}x^8$ (the Square of the Term $-23x^4$ multiply'd by the Fraction over its Head $\frac{17}{35}$) is leſs than $15x^5\times18x^3-\overline{-5x^6\times10x^2}+\overline{x^7\times-28x}=292x^8$, I place the Sign — under the Term $23x^4$. Becauſe $\overline{18x^3}|^2\times\frac{17}{35}$ or $\frac{5508}{35}x^6$ exceeds $\overline{-23x^4\times10x^2}-\overline{15x^5\times-28x}+\overline{-5x^6\times24}=70x^6$ I place the Sign + under the Term $18x^3$. Since $\overline{10x^2}|^2\times\frac{10}{21}$ or $\frac{1000}{21}x^4$ is leſs than $\overline{+18x^3\times-28x}-\overline{-23x^4\times24}=48x^4$ I place the Sign — under the Term $10x^2$. Becauſe $\overline{28x}|^2 \times \frac{3}{7}$ or $336x^2$ is greater than $10x^2 \times 24 = 240x^2$ under $28x$ I place $+$, then under the firſt and laſt Terms I place $+$; and the ſix Changes of under-written Signs ſhews that there are ſix impoſſible Roots.

If the impoſſible Roots were to be found by the *Newtonian* Rule, the Operation would ſtand thus:

$$\underset{+}{x^7} - \underset{-}{\overset{\frac{3}{7}}{5x^6}} + \underset{+}{\overset{\frac{5}{9}}{15x^5}} - \underset{+}{\overset{\frac{3}{4}}{23x^4}} + \underset{+}{\overset{\frac{3}{3}}{18x^3}} + \underset{+}{\overset{\frac{5}{9}}{10x^2}} - \underset{+}{\overset{\frac{3}{7}}{28x}} + \underset{+}{24} = 0,$$

by which Rule there are found only two impoſſible Roots, whereas there are ſix to wit $1 + \sqrt{-3}$, $1 - \sqrt{-3}$, $1 + \sqrt{-2}$, $1 - \sqrt{-2}$, $1 + \sqrt{-1}$, $1 + \sqrt{-1}$, the ſeventh Root being -1.